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A COMPARISON OF FREQUENCY DOMAIN DESIGN AND l^1–OPTIMAL CONTROL

by

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ABSTRACT

A frequency domain design methodology is applied to a DC motor-speed control system and the results are compared to those obtained using l^1–optimal control theory (Pearson and Bamieh [1990]). Both methods synthesize controllers that maximize the allowable size of an unknown-but-bounded disturbance while satisfying prespecified constraints on the control, control rate and the outputs. The frequency domain design technique in general results in much lower order compensators than those required by the l^1–optimal method for a given size of disturbance. Also, the design trade-offs regarding the bandwidth of the system, the size of disturbance input, and the structural complexity of the controller transfer function become quite transparent.

Key Words– Frequency domain design; l^1–optimal control; QFT.

I. INTRODUCTION

The controller design technique given in Jayasuriya and Franchek [1988] synthesizes controllers for maximizing the allowable size of persistent bounded disturbances while keeping the control, the control rate and the outputs of the system within prespecified bounds.

The basic philosophy of the design procedure is to synthesize the disturbance rejection system based solely on the size of a step input disturbance and parallels the Quantitative Feedback Theory (QFT) of Horowitz [1963]. However, under appropriate conditions, any persistent bounded disturbance limited by the size of the step also satisfies the time domain constraints on the control, the control rate, and the outputs.

The design procedure is as follows. First, a set of target transfer functions are identified by mapping the time domain constraints into the frequency domain. Then, the resulting target transfer functions are utilized to determine, at each frequency, an allowable region in the complex s-plane within which the loop transfer function must lie. Finally, classical loop shaping is used to realize an acceptable loop transfer function. During loop shaping, it is possible to take into account any bandwidth limitations imposed on the loop transfer function.

A geometric interpretation based on intersections of a set of circles each corresponding to a specific time domain constraint was given in Zentgraf and Jayasuriya [1989]. This interpretation facilitates the determination of the region within which the nominal loop transfer function must lie. Using this approach, it is also possible to estimate or (under certain conditions) compute exactly the maximum size of the persistent disturbance the system can potentially tolerate. Once a controller is designed, the true maximum size of the persistent disturbance that the designed control system is able to reject can be easily determined.

The paper is organized as follows. In section II, the design methodology is reviewed. In section III, the design technique is applied to a DC motor-speed control system and the results are compared with those obtained using the l^1–optimal method developed by Pearson and Bamieh [1990]. Finally, the conclusions are presented in section IV.

II. DESIGN METHODOLOGY

II.1 Problem Formulation:

Consider the linear time invariant single-input multiple-output (SIMO) system represented by

$$\dot{\mathbf{x}}(t) = \mathbf{A}\ \mathbf{x}(t) + \mathbf{B}\ u(t) + \mathbf{G}\ d(t) \tag{1}$$
$$\mathbf{y}(t) = \mathbf{C}\ \mathbf{x}(t), \tag{2}$$

where $\mathbf{x}(t) \in \mathbf{R}^n$ is the state vector, $u(t) \in \mathbf{R}^1$ is the scalar control, $d(t) \in \mathbf{R}^1$ is the scalar input disturbance and $\mathbf{y}(t) \in \mathbf{R}^m$ is the output vector. The matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{G}$ and $\mathbf{C}$ are of appropriate dimensions.

The output signals $y_k(t)$, $k = 1, 2, \ldots, m$ are bounded by $\beta_k, k = 1, \ldots, m$, the control variable $u(t)$ is bounded by β_u and the control rate $\dot{u}(t)$ is bounded by $\beta_{\dot{u}}$. i.e.,

$$|y_k(t)| \le \beta_k, \qquad k = 1, \ldots, m \tag{3}$$
$$|u(t)| \le \beta_u \tag{4}$$
$$|\dot{u}(t)| \le \beta_{\dot{u}}, \tag{5}$$

for $t \in [0, \infty)$. The objective is to synthesize a controller that satisfies the time domain constraints while maximizing the size of the acceptable bounded disturbance signal, i.e., minimize $\gamma > 0$, where $|d(t)| \le \frac{1}{\gamma}$.

II.2 Outputs and Control Transfer Functions:

Taking the Laplace transform of (1) and (2) gives

$$Y_k(s) = G_{ku}(s)\ U(s) + G_{kd}(s)\ D(s), \tag{6}$$

where

$$G_{ku} = [\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}]_k$$
$$G_{kd} = [\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{G}]_k,$$

and $k = 1, 2, \ldots, m$.

As shown in Zentgraf and Jayasuriya [1989], any output can be chosen for feedback without altering the maximum allowable

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size of the input disturbance. Some outputs may, however, be preferable for feedback than others. It is typical to use an output that generates design boundaries leading to a simple loop shaping problem.

Therefore, without loss of any generality, by choosing y_i as the output to be fed back, the block diagram shown in Fig. 1 can be drawn.

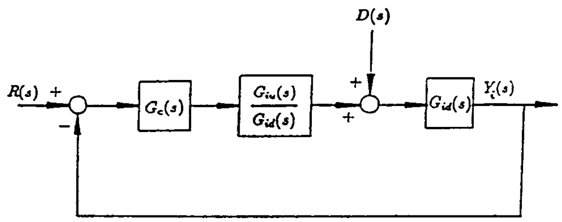

Figure 1: Block Diagram of the Feedback Control System

From Fig. 1 we can write

$$\frac{Y_i(s)}{D(s)} = \frac{G_{id}(s)}{1+L_o(s)} \tag{7}$$

$$\frac{U(s)}{D(s)} = \frac{-G_{id}(s)}{G_{iu}(s)} \cdot \frac{L_o(s)}{1+L_o(s)} \tag{8}$$

$$\frac{\dot{U}(s)}{D(s)} = \frac{-s \cdot G_{id}(s)}{G_{iu}(s)} \cdot \frac{L_o(s)}{1+L_o(s)}, \tag{9}$$

where

$$L_o(s) = G_c(s) \cdot G_{iu}(s) \tag{10}$$

is the loop transfer function.

From (6) and (8), the transfer functions relating the outputs to the disturbance input are

$$\frac{Y_k(s)}{D(s)} = G_{kd}(s) \left[1 + \frac{-G_{ku}(s) \cdot G_{id}(s)}{G_{iu}(s) \cdot G_{kd}(s)} \cdot \frac{L_o(s)}{1+L_o(s)}\right], \tag{11}$$

where $k = 1, 2, \ldots, m$.

II.3 Generating Frequency Domain Bounds:

The frequency domain design boundaries are obtained by mapping the time domain constraints into their frequency domain equivalents. To accomplish this, the following lemma (Jayasuriya [1990]) is used.

Lemma: A stable transfer function $G(s)$ with $|G(jw)| \leq M$, has a unit step response bounded as:

$$\left|\mathcal{L}^{-1}\left\{G(s) \cdot \frac{1}{s}\right\}\right| \leq 4M.$$

For a step size other than unity, we can scale the frequency response by γ, giving

$$|G(j\omega)| \leq \gamma M,$$

which in turn leads to a step size of $\frac{1}{4\gamma}$, yielding a step response upper bounded by M.

With the above result, we can now generate frequency domain design boundaries that assure the satisfaction of the time domain constraints. Therefore, from Eqns. (3)- (5) we obtain the following:

$$\gamma \cdot \beta_u \geq \left|\frac{U(j\omega)}{D(j\omega)}\right| \tag{12}$$

$$\gamma \cdot \beta_{\dot{u}} \geq \left|\frac{(j\omega) \cdot U(j\omega)}{D(j\omega)}\right| \tag{13}$$

$$\gamma \cdot \beta_k \geq \left|\frac{Y_k(j\omega)}{D(j\omega)}\right|, \quad k = 1, 2, \ldots, m, \tag{14}$$

where $\gamma = \frac{1}{\alpha}$ with α being the size of the step disturbance.

Substituting (8) - (11) into (12) - (14) and rearranging terms gives

$$|L_c(j\omega)| \leq \gamma\beta_u \left|\frac{G_{iu}(j\omega)}{G_{id}(j\omega)}\right| \tag{15}$$

$$|L_c(j\omega)| \leq \gamma\beta_{\dot{u}} \left|\frac{(j\omega)G_{iu}(j\omega)}{G_{id}(j\omega)}\right| \tag{16}$$

$$\left|L_c(j\omega) - \frac{G_{iu}(j\omega)G_{kd}(j\omega)}{G_{ku}(j\omega)G_{id}(j\omega)}\right| \leq \gamma\beta_k \left|\frac{G_{iu}(j\omega)}{G_{ku}(j\omega)G_{id}(j\omega)}\right|, \tag{17}$$

where $k = 1, 2, \ldots, m$ and $L_c(j\omega) = \frac{L_o(j\omega)}{1+L_o(j\omega)}$.

Note that inequalities (15) – (17) are of the form:

$$|L_c(j\omega) - C(j\omega)| \leq |R(j\omega)|, \tag{18}$$

with $L_c(j\omega)$ lying in the closed region enclosed by a circle centered at $C(j\omega)$ and radius $|R(j\omega)|$ at each frequency, ω. Thus the common intersection of these circles defines the allowed region within which the desired closed loop transfer function, $L_c(j\omega)$, should lie at a specific frequency (Fig. 2).

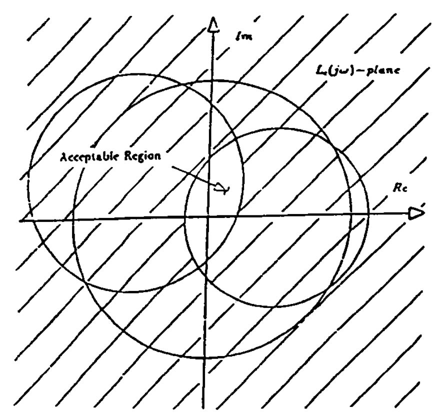

Figure 2: Allowed Region for a Closed Loop Transfer Function

Determination of the estimated minimum value of $\gamma = \frac{1}{\alpha}$, which corresponds to the maximum allowed size of the step disturbance, now becomes a max-min problem. i.e., at each frequency we determine a minimum for $\gamma(\omega)$ which collapses the allowed region defined by (15)–(17) to a point, and then choose the maximum of γ over all ω's. Hence, the estimated maximum step disturbance magnitude over the frequencies $\omega \in \Omega = [0, \infty)$ is given by $\alpha^{max} = \frac{1}{\gamma^{min}}$ where $\gamma^{min} = \sup_{\omega \in \Omega} \gamma$ (Fig. 3).

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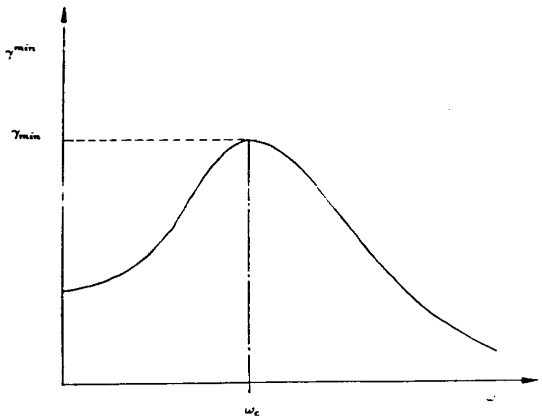

Figure 3: Optimal Values of γ vs. Frequency

Substituting γ^{min} for γ in (15)-(17), we obtain the allowed regions in the complex s-plane within which the loop transfer function must lie. The final loop shaping can be carried out by transferring these allowed regions into a Nichols chart (Fig. 4) over the frequency range of interest.

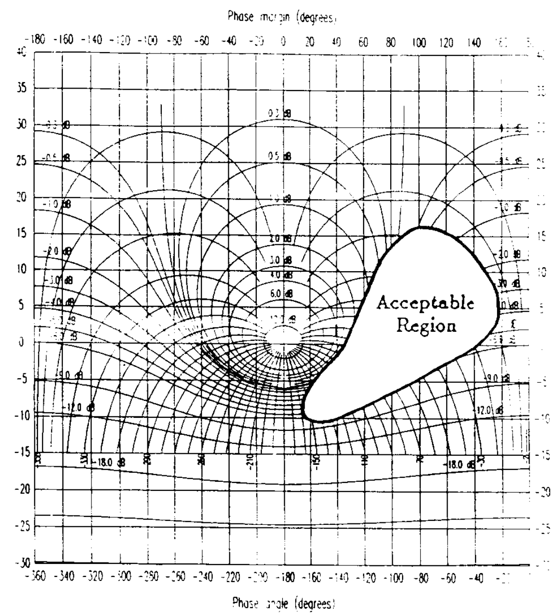

Figure 4: Frequency Domain Regions Plotted on a Nichols Chart

II.4 Size of the Maximum Step Disturbance:

Once a loop transfer function $L_o(s)$ is designed, the following bounds on the tolerable size of the step disturbance can be calculated (see Jayasuriya and Sobhani [1991] for complete proofs.)

Proposition 1: If the unit step response of a stable transfer function $G(s)$ is bounded by ν then $|G(j0)| \leq \nu$.

Theorem 1: If the maximum value of $\gamma(\omega)$, i.e. γ^{min}, occurs at zero frequency, then the maximum tolerable size of a step disturbance is $\alpha^{max} = \frac{1}{\gamma^{min}}$.

If γ^{min} occurs at a frequency other than zero, the maximum size of the step disturbance that the system can tolerate is determined by obtaining the unit step disturbance response of the closed loop system and noting the maximum values of u, $\dot{u}$ and y_k, denoted by u^{max}, $\dot{u}^{max}$ and y_k^{max}, $k = 1, \ldots, m$. The maximum step disturbance size is then given by

$$min\left\{\frac{\beta_u}{|u^{max}|}; \frac{\beta_{\dot{u}}}{|\dot{u}^{max}|}; \frac{\beta_k}{|y_k^{max}|}, k = 1, \ldots, m\right\}.$$

II.5 Size of the Maximum Persistent Disturbance:

The size of the maximum persistent disturbance that the controlled system can reject is computed as follows (Jayasuriya [1989]):

Assume $\frac{Y(s)}{D(s)} = G(s)$ is strictly proper with $|y(t)| \leq \beta$ then

$$\max|d(t)| = \frac{\beta}{\int_0^t |g(t-\tau)|d\tau}. \qquad (19)$$

Therefore, if the impulse response of the system does not change sign, then the size of the maximum step disturbance will be equal to the size of the maximum persistent disturbance, since $|g(t-\tau)| = g(t-\tau)$. That is

$$\max|d(t)| = \frac{\beta}{\int_0^t g(t-\tau)d\tau}. \qquad (20)$$

However, if the impulse response changes its sign then one can always find the worst disturbance input $d(t)$ in the form of a "square wave". The worst disturbance corresponds to an input that makes $g(t-\tau) \cdot d(\tau)$ always positive. In that case, the maximum persistent disturbance would be less than the size of the maximum step disturbance.

III. APPLICATION TO A DC MOTOR - SPEED CONTROL PROBLEM

In this section, the design technique is applied to a DC motor-speed control problem and the results are compared with those obtained by the l^1-optimization method (Pearson and Bamieh [1990].) The block diagram of the system is shown in Fig. 5.

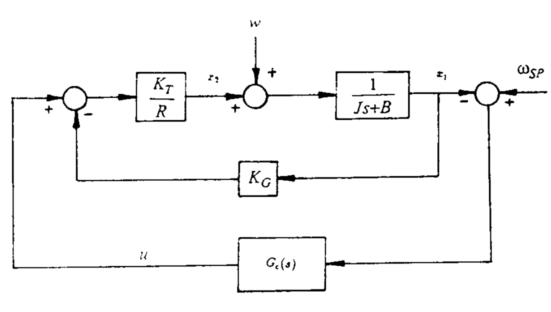

Figure 5: Block Diagram of the DC Motor-Speed Control System

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Assuming unity values for all the constants (K_T, K_G, J, B, R), the following transfer functions are derived:

$$\mathbf{X}(s) = \mathbf{G_u}(s) \cdot U(s) + \mathbf{G_w}(s) \cdot W(s), \tag{21}$$

where

$$\mathbf{X}(s) = \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} \tag{22}$$

$$\mathbf{G_u}(s) = \begin{bmatrix} G_{1u}(s) \\ G_{2u}(s) \end{bmatrix} \tag{23}$$

$$\mathbf{G_w}(s) = \begin{bmatrix} G_{1w}(s) \\ G_{2w}(s) \end{bmatrix} \tag{24}$$

where $U(s) = \mathcal{L}[u(t)]$ is the controller variable, $W(s) = \mathcal{L}[w(t)]$ the disturbance, $X_1(s) = \mathcal{L}[x_1(t)]$ and $X_2(s) = \mathcal{L}[x_2(t)]$ the states. The transfer functions $\mathbf{G_u}(s)$ and $\mathbf{G_w}(s)$ are given as

$$\begin{aligned} \mathbf{G_u}(s) &= \tfrac{1}{\Delta(s)} \cdot \begin{bmatrix} 1 \\ 1+s \end{bmatrix} \\ \mathbf{G_w}(s) &= \tfrac{1}{\Delta(s)} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \end{aligned} \tag{25}$$

where $\Delta(s) = s + 2$.

The objective is to find a controller that keeps for all times $t \in [0, \infty)$ the following bounds on the state $x_2(t)$, the controller $u(t)$ and its rate $\dot{u}(t)$ while maximizing the size of the allowable disturbance $w(t)$:

$$|x_2(t)| \le 1 \tag{26}$$

$$|u(t)| \le 1 \tag{27}$$

$$|\dot{u}(t)| \le 1. \tag{28}$$

The l^1–optimal control gives the following results (Pearson and Bamieh [1990])

$n = 8$	$\gamma = 1.562587$
$n = 10$	$\gamma = 0.917115$
$n = 20$	$\gamma = 0.465110$
$n = 40$	$\gamma = 0.341940$
$n = 60$	$\gamma = 0.333394$
$n = 80$	$\gamma = 0.333335$
$n = 100$	$\gamma = 0.333333$

where n is the degree of the polynomial matrix relating the constrained variables to the disturbance and γ is the reciprocal of the maximum allowable size of the persistent disturbance. Therefore, to reject a disturbance of magnitude 3.00, an approximately fiftieth order compensator is needed.

In the following, we apply our methodology and show that the same maximum size of the disturbance can be determined and rejected by much lower order loop transfer functions.

Since bandwidth was no subject of concern, x_1 is arbitrarily chosen for feedback. Therefore, using inequalities (15)–(17), the following frequency domain bounds are obtained.

For State 2:

$$\left| L_c(j\omega) - \frac{-1}{s+1} \right| \le \frac{1}{\gamma} \cdot \left| \frac{s+2}{s+1} \right| \tag{29}$$

For Control:

$$|L_c(j\omega)| \le \frac{1}{\gamma} \tag{30}$$

For Rate of Control:

$$|L_c(j\omega)| \le \frac{1}{\gamma} \cdot |s|. \tag{31}$$

Using the inequalities (29) – (31), a closed loop circle family can be determined whose common intersection area displays the necessary location for any valid closed loop transfer function $L_c(s)$. The optimal value for γ was found for $\omega \to 0$ with $\gamma^{min} = \frac{1}{3}$ equivalent to a disturbance magnitude 3.00, for that the area is shrunk to a single point (Fig. 6). Note that since γ^{min} occurs at zero frequency, the maximum size of the disturbance is indeed equal to 3.00 (which is the same as the l^1–optimal result.)

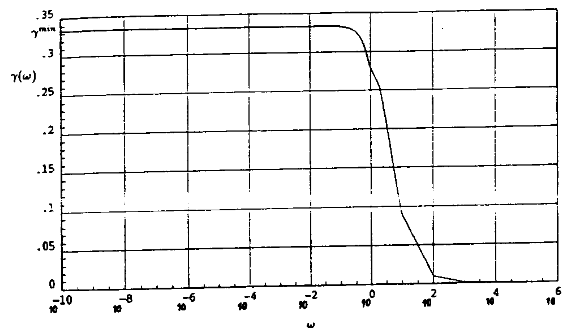

Figure 6: Optimal Values of γ vs. Frequency for the Example

Substituting $\gamma = \frac{1}{3}$ in inequalities (29) – (31), the frequency domain allowed regions for the loop transfer function can be determined on a Nichols chart shown in Fig. 7.

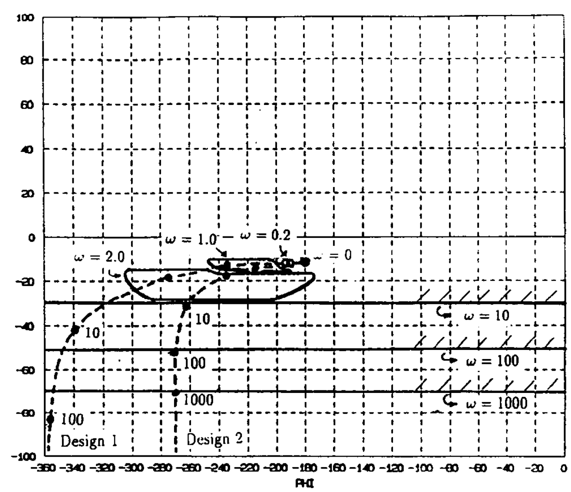

Figure 7: Frequency Domain Bounds on $L_o(s)$ for $(\alpha = 3.00)$

A loop transfer function $L_o(s)$ that meets magnitude and phase bounds for $\gamma = \frac{1}{3}$ is

$$L_o(s) = -\frac{1}{3} \cdot \frac{1}{s + \frac{4}{3}}, \tag{32}$$

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yielding the following controller:

$$G_c(s) = -\frac{1}{3} \cdot \frac{s+2}{s+\frac{4}{3}}. \qquad (33)$$

Step responses of x_2, u and $\dot{u}$ are shown in Fig. 8 which illustrates the satisfaction of time domain constraints for step disturbance of magnitude 3.00. As shown in Fig. 9, the impulse responses of the constrained variables do not change sign. Therefore, the maximum allowable size of the persistent disturbance is also 3.00.

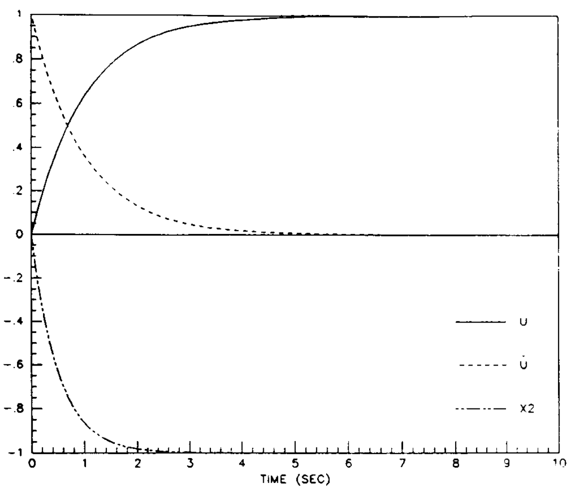

Figure 8: Time Responses for Step Disturbance (α = 3.00) - Design 1

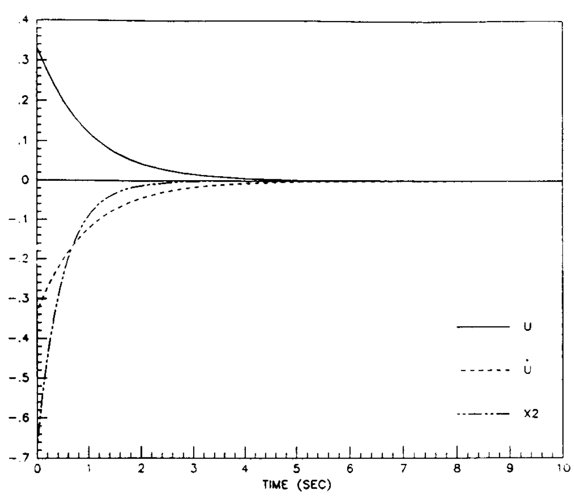

Figure 9: Impulse Responses - Design 1

Although the controller transfer function $G_c(s)$ results in satisfaction of all time domain constraints, it is not strictly proper and therefore has an infinite bandwidth (which is not practically feasible). However, any attempt to roll off the loop transfer function at some high frequency range by adding a far-off pole to $L_o(s)$ will cause the impulse response of $\dot{u}$ to change sign which leads to a reduction of the size of the maximum persistent disturbance. This is transparent from the fact that both u and $\dot{u}$ are forced to have impulse responses of one sign. However, it is difficult if not impossible to obtain a strictly proper controller transfer function $G_c(s)$ unless the controller transfer function is made to have a more complex structure needed to approximate the behavior of a non-strictly proper transfer function, i.e. a higher order controller is needed. Note that the need for high order controllers is consistent with the results from l^1–optimization method. However, as will be shown in the following, the order of the controller need not be as high as the one reported for the l^1–optimal approach. A much lower order controller transfer function can yield acceptable results as discussed below.

The following loop transfer function is designed:

$$L_o(s) = -0.825 \cdot \frac{1}{(s+2)(s+1.65)}, \qquad (34)$$

which yields the following strictly proper controller transfer function:

$$G_c(s) = -0.825 \cdot \frac{1}{s+1.625}. \qquad (35)$$

The frequency responses of $L_o(s)$ and $G_c(s)$ in (34) and (35) are shown in Figs. (10) and (11), respectively. The time responses of x_2, u and $\dot{u}$ for a step disturbance of magnitude 3.00 are given in Fig. 12, which shows the satisfaction of the time domain constraints. However, as can be seen from Fig. 13, the impulse response of $\dot{u}$ changes sign and has an absolute area of 0.3526 which yields the maximum size of the persistent disturbance $= \frac{1}{0.3526} = 2.84 < 3.00$. Note that for rejecting this size of the persistent disturbance, the l^1–optimization technique calls for a transfer function of at least $30th$ order.

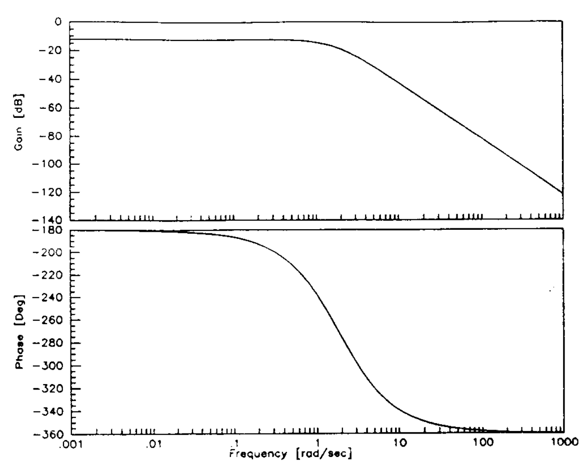

Figure 10: Frequency Response of $L_o(s)$ - Design 2

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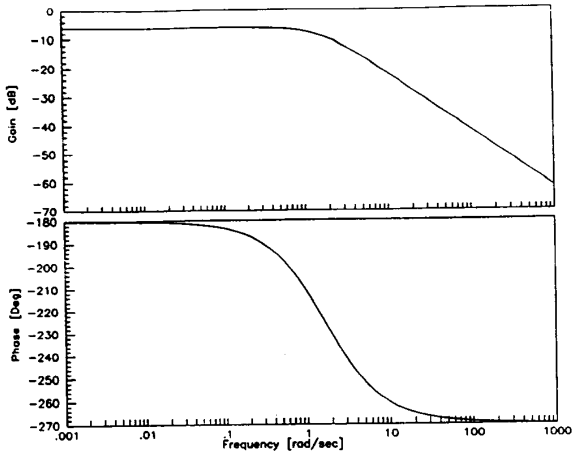

Figure 11: Frequency Response of $G_c(s)$ - Design 2

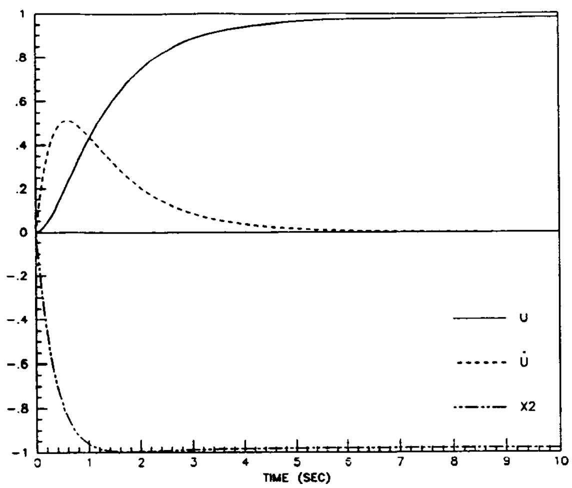

Figure 12: Time Responses for Step Disturbance ($\alpha = 3.00$) - Design 2

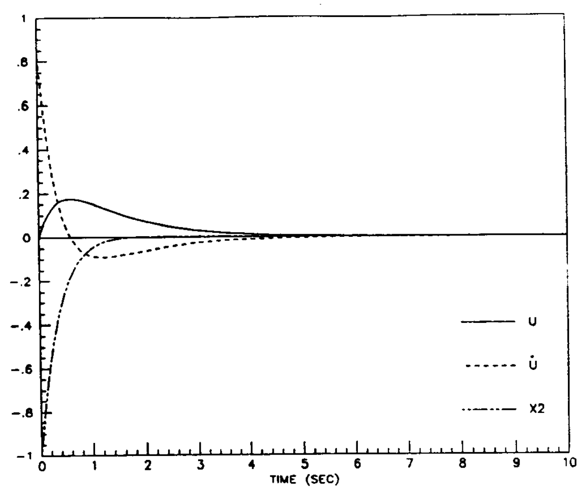

Figure 13: Impulse Responses - Design 2

IV. CONCLUSIONS

A frequency domain design methodology for synthesizing controllers that keep the outputs, the control, and the control rate of a system within pre-specified bounds while maximizing the size of a persistent disturbance input, was reviewed. Application of the design methodology to a DC motor-speed control system was analyzed and the results were compared with those obtained by the l^1–optimal technique. The frequency domain design technique provides important insights to the design trade-offs such as the maximum allowable size of the disturbance, the bandwidth considerations, and the structural complexity of the controller transfer function. The design technique can achieve the same maximum tolerable disturbance as those obtained by l^1–method with much lower order controllers. A combination of l^1–theory with the frequency domain technique may provide meaningful solution to the class of problems studied. Research is underway to establish and use such connections.

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